Written Exam for the M.Sc. in Economics Autumn 2015 (Fall Term)

Financial Econometrics A: Volatility Modelling

Final Exam: Masters course

Exam date: **January 5, 2016**

3-hour closed book exam.

Please note there are a total of **8** questions which should **all be replied to**. That is, **4** questions under *Question A*, and **4** under *Question B*.

Total numbers of pages (including this one): 5

Please also note that the language used in your exam paper must correspond to the language of the title for which you registered during exam registration. I.e. if you registered for the English title of the course, you must write your exam paper in English. Likewise, if you registered for the Danish title of the course or if you registered for the English title which was followed by "eksamen på dansk" in brackets, you must write your exam paper in Danish. If you are in doubt about which title you registered for, please see the print of your exam registration from the students' self-service system.

Question A:

In finance (as well as elsewhere) much attention has recently been on OLS regression of returns y_t on predictive regressors x_t. We can write this as,

$$y_t = \beta x_t + \varepsilon_t, \qquad t = 1, 2, ..., T. \tag{1}$$

In the next we will make different assumptions about the stochastic properties of the innovations ε_t. The interest lies in testing if $\beta = 0$, which we write as $H : \beta = \beta_0 = 0$.

It is well-known that if ε_t are i.i.d.N$(0, 1)$ distributed the MLE of β is given by the OLS estimator,

$$\hat{\beta} = \sum_{t=1}^{T} y_t x_t \left(\sum_{t=1}^{T} x_t^2 \right)^{-1}. \tag{2}$$

Question A.1: When $\beta_0 = 0$ (that is, the hypothesis H holds) it follows that,

$$\hat{\beta} = \frac{1}{T} \sum_{t=1}^{T} \varepsilon_t x_t \left(\frac{1}{T} \sum_{t=1}^{T} x_t^2 \right)^{-1} \tag{3}$$

Discuss conditions on x_t which ensure

$$\frac{1}{T} \sum_{t=1}^{T} x_t^2 \xrightarrow{p} \sigma_x^2 > 0. \tag{4}$$

Discuss briefly one or two examples of financial variables which may satisfy this condition. Explain why.

Question A.2: Assume next that ε_t follows an ARCH specification, that is ε_t is an ARCH(1) process given by,

$$\varepsilon_t = \sigma_t \eta_t, \quad \eta_t \text{ i.i.d.N}\,(0, 1) \quad \text{and} \quad \sigma_t^2 = 1 + \alpha \varepsilon_{t-1}^2. \tag{5}$$

Under which conditions on α does it hold that

$$\frac{1}{T} \sum_{t=1}^{T} \varepsilon_t^4 \xrightarrow{p} \kappa_4 > 0? \tag{6}$$

Be specific about why it holds. (*Hint*: Recall that $E[\eta_t^4] = 3$)

Question A.3: Next assume that ε_t follow an ARCH specification and that $(\varepsilon_t, x_t)'$ is a Markov chain which satisfies a drift criterion with drift function,

$$\delta(\varepsilon, x) = 1 + \varepsilon^4 + x^4. \tag{7}$$

Show that under this assumption and that $E[\eta_t x_t | \varepsilon_{t-1}, x_{t-1}] = 0$,

$$\sqrt{T}\hat{\beta} = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \varepsilon_t x_t \left(\frac{1}{T} \sum_{t=1}^{T} x_t^2 \right)^{-1} \xrightarrow{d} N\left(0, \sigma_{\varepsilon x}^2 / \left(\sigma_x^2\right)^2\right) \tag{8}$$

where $\sigma_{\varepsilon x}^2 = E\left(\varepsilon_t^2 x_t^2\right)$ is finite.

Question A.4: By Question A.3 one can conclude that despite ARCH effects the estimator of the OLS estimator $\hat{\beta}$ is still asymptotically Gaussian. Explain how you would find the MLE of β when ε_t is given by the ARCH(1) model.

Question B:

Suppose that the efficient log-price of a share of stock at time t is given by

$$P(t) = \sigma W(t), \quad t \in [0, T],$$

where $W(t)$ is a Brownian motion, $\sigma > 0$ is constant, and $T > 0$.

Question B.1: What is the distribution of $P(t)$?
With $t - 1 \geq 0$, let

$$r(t) = P(t) - P(t-1).$$

What is the mean and variance of $r(t)$?

Question B.2: Suppose that the price P is observed at $n+1$ equidistant points between time t and $t-1 \geq 0$, that is we observe $\{P(t_i) : i = 0, 1, ..., n\}$ where $t_i = t - 1 + i/n$.
One way to measure the volatility of $r(t)$ would be to compute the realized volatility given by

$$RV(t, n) = \sum_{i=1}^{n} \left(P(t_i) - P(t_{i-1})\right)^2.$$

Use that

$$P(t_i) - P(t_{i-1}) = \sigma(W(t_i) - W(t_{i-1}))$$

in order to find the probability limit of $RV(t, n)$ as $n \to \infty$. Be precise about the arguments used for deriving the probability limit.
Give an interpretation of letting $n \to \infty$.

Question B.3: Suppose that we do not observe the efficient price $P(t)$, but instead we observe $\tilde{P}(t)$ which is $P(t)$ contaminated by some noise $\tilde{\varepsilon}(t)$, that is

$$\tilde{P}(t) = P(t) + \tilde{\varepsilon}(t), \quad t \in [0, T],$$

with

$$\tilde{\varepsilon}(t) = \tilde{\sigma}\tilde{W}(t),$$

where $\tilde{W}(t)$ is a Brownian motion and $\tilde{\sigma} > 0$ is constant.
Now the realized volatility measure $RV(t, n)$ from the previous question is infeasible due the fact that we do not observe $P(t)$. Instead we may compute

$$\widetilde{RV}(t, n) = \sum_{i=1}^{n} \left(\tilde{P}(t_i) - \tilde{P}(t_{i-1})\right)^2.$$

Assume that $W(t)$ and $\tilde{W}(t)$ are independent, that is $(W(t) : t \in [0,T])$ and $(\tilde{W}(t) : t \in [0,T])$ are independent. Similar to the previous question, derive the probability limit of $\widetilde{RV}(t,n)$ as $n \to \infty$. Compare with the probability limit of $RV(t,n)$.

Question B.4: Figure 1 contains a plot of the realized volatility of the return of the Euro/Dollar exchange rate. At day $t = 1, 2, ..., 796$ the realized volatility is based on $n = 47$ intra-daily return observations. Based on the figure and in light of your findings in the previous questions, do you think that the model $P(t) = \sigma W(t)$, from Question B.1 is suitable for the log-price of the exchange rate? Discuss briefly.

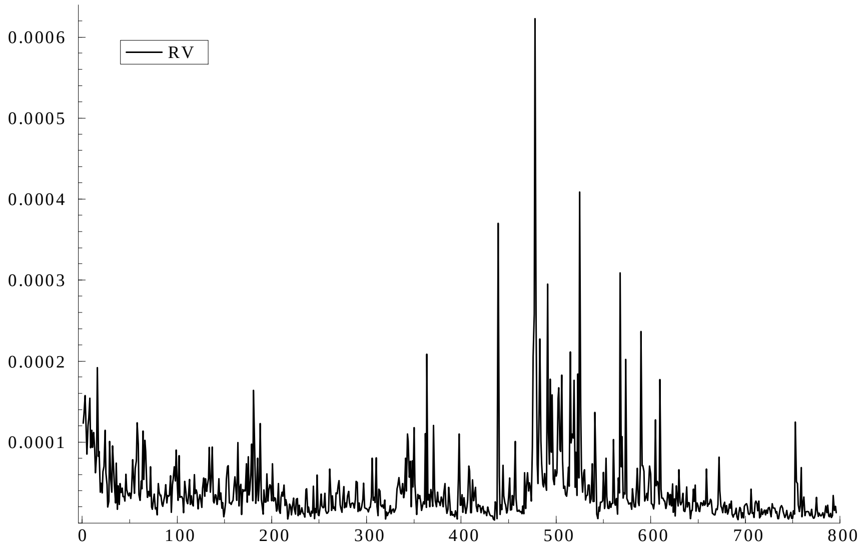

Figure 1: RV of Euro/Dollar returns